4D Gauge Theory Reading Seminar

I'm helping to organise a reading seminar for MT 2022. The idea is to see a proof of Donaldson's theorem and compare the theory of ASD instantons with Seiberg-Witten theory, the latter being easier but the former being more relevant to gauge theory in higher dimensions.

The rough schedule for the talks will be

1. Definitions, introduction to Gauge theory and constructing manifold invariants.
2. Yang Mills functional, ASD equations. Chern-Weil theory. U(1) example.
3. Spin and Spin^c structures. Dirac Operators. Seiberg-Witten equations.
4. ADHM construction of Instantons on $\mathbb{R}^4$. Compactness problems.
5. Fredholm Theory, Moduli spaces, Uhlenbeck's theorem and Gauge fixing.
6. Intersection forms and Donaldson's Theorem.
7. Overview of the proof of Donaldson's Theorem.
8. Gauge Theoretic Invariants in Algebraic Geometry and Symplectic Geometry.

I plan to write notes after each talk and add them to this page. If you find any mistakes below, please let me know.

1 Introdution to Gauge Theory and Constructing Manifold Invariants.

Alfred Holmes

1.1 Introduction

The goal of this reading group is to give a self contained overview of gauge theory in 4 dimensions. The talks will gover both the theory of ASD instantons and the Seiberg-Witten (SW) equations. The aim of the reading group it to see a proof of Donaldson's theorem on the possible intersection forms of smooth four-manifolds, and to compare the theories of ASD instantons and the SW equations.

In this first talk we will review the basic setup of gauge theory – principal bundles and the action of the gauge group on various forms – and the general procedure on setup can be used to construct manifold invariants. In the first talk there will be nothing specific to dimension 4. This talk is largely based on the lecture notes Haydys - Introduction to Gauge Theory, and is mostly just definitions.

1.2 Principal Bundles, Connections and Curvature

Throughout this talk we fix a manifold M. This is the manifold that we want to define the invariants of.

Definition 1.1. Let G be a Lie group. A principal G-bundle is a smooth manifold P with a smooth (right) G action, such that

$$P/G = M,$$

and P is locally trivial. That is, for each $m \in M$, there is an open set U containing m such that

$$\varphi : P|_U \to U \times G,$$

which is G equivariant, so $\varphi(pg) = \varphi(p)g$, $\forall g \in G$. We say that P has *structure group G*.

Definition 1.2. Given a principal G-bundle $P \to M$, the gauge group, $\mathcal{G}$, of M is the set of G equivariant bundle isomorphisms. That is

$$\mathcal{G} = \{\varphi : P \to P | \varphi(pg) = pg\}$$

Remark 1.3. Let $f : P \to G$ be such that

$$\varphi(p) = pf(p).$$

Then we have that

$$\varphi(pg) = \varphi(p)g = pf(p)g = pgg^{-1}f(p)g$$

and so $f(pg) = g^{-1}f(p)g$. Hence, if $g \in Z(G)$, then the map $p \mapsto pg$ defines a gauge transformation. In physics terminology, this is called a global symmetry - an example of this is a change of phase in the theory of electromagnetism. A local symmetry in physics is just a regular gauge transformation.

We now study the tangent space of a principal bundle $P \to M$. This leads to the notion of a vertical subspace and connections.

Definition 1.4. Let $\phi_p : \mathfrak{g} \to T_pP$ be defined by

$$\xi \mapsto \frac{d}{dt}\, p \exp(t\xi)|_{t=0} \tag{1}$$

Taking the union of the image for each $p \in P$ defines a subbundle with fibres isomorphic to $\mathfrak{g}$. This is the *vertical subspace*.

We now define the vector bundles associated to representations of the structure group. An alternative way of setting up gauge theory is to start with some vector bundle and consider the frame bundle of that vector bundle. In this case the two descriptions agree.

Definition 1.5. Let $P \to M$ be a principal G-bundle and let $\rho : G \to GL(V)$ be a G-representation. Then the associated vector bundle is the quotient

$$E = P \times_\rho G = P \times V/G, \text{ where } (p, v).g = (pg, \rho(g^{-1}v)).$$

Let $\varphi \in \mathcal{G}$. Then this induces a bundle isomorphism $\phi : E \to E$, given by

$$\phi([p, v]) = ([\varphi(p), v]).$$

Note that $\phi([pg, \rho(g^{-1})v]) = ([\varphi(pg), \rho(g^{-1})v]) = ([\varphi(p)g, \rho^{-1}v]) = ([\varphi(p), v])$, so this is well defined.

We now define a bijection between certain sections of $P \times V$, the trivial V bundle over P and sections of the associated bundle E.

Proposition 1.6. Let $s \in \Gamma(E)$. Then there exists an $\hat{s} \in \Gamma(P \times V)$, such that

$$s(\pi(p)) = [p, \hat{s}(p)] \tag{2}$$

for all $p \in P$. More over this defines a bijection $\Gamma(E) \cong \Gamma(P \times V)^G$.

Proof. Note that for $\hat{s} \in \Gamma(P \times V)$ to satisfy (2), we must have that $\hat{s}(pg) = \rho(g^{-1})s(p)$, that is $\hat{s} \in \Gamma(P \times V)^G$. If we pick $p \in \pi^{-1}(m)$, then by definition

$$s(m) = [p, v]$$

for some unique $v \in V$, since $s(m) \in E$. Hence we can define $\hat{s}(p) = (p, v)$. One the other hand, if $\hat{s} \in \Gamma(P \times V)^G$, then $s(m) = [p, \hat{s}(p)]$ is well defined. $\square$

Remark 1.7. We may view gauge transformations as sections of the associated bundle $G \times_C G$, where

$$C : G \to \mathrm{Aut}(G) \tag{3}$$

is conjugation.

We can make a similar identification of forms on P that take values in the vector bundle V. For this we need a further definition.

Definition 1.8. A form $\hat{\omega} \in \Omega^q(P, V)$ taking values in the trivial V bundle over P is basic if $i_{\phi_p \xi}\omega(p) = 0$ for all $\xi \in \mathfrak{g}$, $p \in P$. Here ϕ_p is as in (1).

Definition 1.9. A form $\hat{\Omega}^1(P, V)$ is equivariant if

$$R_g^* \hat{\omega} = \rho(g^{-1})\hat{\omega}$$

where the left hand side is the pull back by the multiplication by the action of g and the left hand side is applied on the trivial fibre.

Proposition 1.10. There is a bijection between basic equivariant forms $\hat{\omega} \in \Omega^q(P, V)$ and forms in the associated bundle $\Omega^q(M, E)$

Proof. This follows from noting that, given such an $\hat{\omega}$, defining $\omega \in \Omega^q(M, E)$ via

$$\omega(\pi(p))(\pi X_1, \ldots, \pi X_q) = [p, \hat{\omega}(X_1, \ldots, X_q)] \tag{4}$$

is well defined. □

We can now define connections on a principal bundle P. Let $\mathfrak{g}$ denote the Lie algebra of G, considered as a G-representation via the adjoint action.

Definition 1.11. Let $P \to M$ be a principal G-bundle. A connection on P, $A \in \Omega^1(P, \mathfrak{g})$ is a G-equivariant one form taking values in $\mathfrak{g}$ such that

$$A(\phi_p \xi) = \xi$$

for all $\xi \in \mathfrak{g}$, where ϕ_p is as in (1).

Remark 1.12. Note that A_p, viewed as a map $T_p P \to \mathfrak{g}$, has rank $\dim G$ and so A defines a horizontal subspace $H = \ker A$. This is G-invariant ($H.G \subseteq H$) and defining such an H is equivalent to defining A.

Lemma 1.13. The space of all connections is an affine space modelled on $\Omega^1(M, \mathrm{Ad}P)$, where

$$\mathrm{Ad}P = P \times_{Ad} \mathfrak{g}.$$

Proof. The difference of two connections is basic. The result then follows from Proposition 1.10. □

We can consider the action of $\mathcal{G}$ on the space of connections.

Proposition 1.14. Let $\varphi \in \mathcal{G}$. Then if $A \in \Omega^1(P, \})$ is a connection,

$$\varphi^* A = f^{-1} A f + \left(L_{f^{-1}}\right)_* df$$

where $\varphi(p) = pf(p)$ and $L : G \to G$ is left multiplication.

Proof. We can view φ as the composition

$$\begin{aligned} \varphi : P &\xrightarrow{\Delta} P \times G \xrightarrow{R} P \\ p &\mapsto (p, f(p)) \mapsto pf(p) \end{aligned}$$

We have that the pushforward

$$(\Delta_p)_* v = (v, df(v)) \in T(P \times G)_{(p, f(p))}$$

and that

$$(R_{(p,f(p))})_* \left(v, (L_{f(p)})_* \xi\right) = (R_{f(p)})_* v + \phi_{pf(p)} \xi.$$

To see this, note that the pushforward is linear and that

$$(R_{(p,f(p))})_* \left(0, (L_{f(p)})_* \xi\right) = \frac{d}{dt}\, pf(p) \exp(t\xi)|_{t=0}$$

Hence we have

$$\begin{aligned} \varphi^* A(p)(v) = A(pf(p)) \left(R_{(p,f(p))(\Delta_p)_* v}\right) &= A(pf(p)) \left((R_{f(p)})_* v + \phi_{pf(p)} (L_{f^{-1}})_* df(v)\right) \\ &= \left(f^{-1} A(p) f\right)(v) + (L_{f^{-1}})_* df(v) \end{aligned}$$

since A is a connection. □

We can also use A to define a covariant derivative. To do this, define, for $s \in \Gamma(P \times V)$, the operator

$$d_A s = ds + A.s \in \Omega^1(P, V).$$

The action of A on s is given by the derivative of the representation $\rho : G \to GL(V)$. This can be extended to an operator

$$d_A : \Omega^q(P, V) \to \Omega^{q+1}(P, V)$$

in the standard way. We define

$$d_A \omega \wedge \theta = (d_A \omega) \wedge \theta + (-1)^p \omega \wedge d\theta$$

for $\omega \in \Omega^p(P, V)$ and $\theta \in \Omega^q(P)$.

This derivative operator defines a derivative operator on the associated bundles. This follows from the following proposition.

Proposition 1.15. If $\hat{\omega} \in \Omega(P, V)$ is G-equivariant and basic, then $d_A \hat{\omega}$ is also G-equivariant and basic.

Proof. Equivariance of $d_A \hat{\omega}$ follows from the fact that $\hat{\omega}$, A and $d\hat{\omega}$ are G-equivariant. To prove that $d_A \hat{\omega}$ is basic it is sufficient to consider

$$i_\xi (d\hat{\omega} + A.\hat{\omega}) = \pounds_\xi \hat{\omega} + [\xi, \hat{\omega}] = -[\xi, \hat{\omega}] + [\xi, \hat{\omega}].$$

Hence, if one has a vector bundle with a derivative operator, then this defines a connection on the frame bundle. □

Definition 1.16. Let A be a connection. The curvature of A, $F_A \in \Omega^2(\mathfrak{g})$ is defined to be the map

$$F_A : \Gamma(P, \mathfrak{g}) \to \Omega^2(P, \mathfrak{g})$$

given by $F_A s = d_A^2 s$. This is a tensor in the sense that $F_A s$ depends C^∞ linearly on s and we have that under gauge transformations, $\varphi^* F_A = f^{-1} F_A f$.

Proof. We have

$$\begin{aligned} d_A d_A s &= d^2 s + d(A.s) + A.\left(ds + A.s\right) \\ &= dA \wedge s + A \wedge ds + A \wedge ds + A \wedge A.s \\ &= \left(dA + A \wedge A\right).s. \end{aligned}$$

Note that we can view the values of $dA + A \wedge A$ as elements of $\mathfrak{g}$, under the image $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. □

1.3 Constructing Manifold Invariants

We now conclude by considering how the above machinery can be used to construct manifold invariants. In this set up we consider an infinite dimensional manifold $\mathcal{C}$ (for example, the space of connections, or a connection with some other data) and consider an action by a lie group $\mathcal{G}$ (for example the gauge group) on the right. We then want to consider the quotient

$$\mathcal{B} = \mathcal{C}/\mathcal{G}.$$

We want this to be a nice space, in particular another manifold. Hence we may want to restrict to $\mathcal{C}^* \subseteq \mathcal{C}$ where the action of $\mathcal{G}$ on $\mathcal{C}^*$ is free, and consider the quotient $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$.

We now pick a representation V of $\mathcal{G}$. This is, for example, $\Omega^2(P, \mathfrak{g})$, where the action of the gauge group $\mathcal{G}$ is given by conjugation. Our gauge theoretic equations are then an $\mathcal{G}$-equivariant map

$$F : \mathcal{C} \to V.$$

For the ASD equations, this is just given by taking the self dual part of the curvature.

To this choice $(\mathcal{G}, \mathcal{C}, V, F)$, we can associate a moduli space $\mathcal{M}$, which we define to be

$$\mathcal{M} = (F^{-1}(0) \cap \mathcal{C}^*)/\mathcal{G}.$$

We hope that this is a finite dimensional manifold. This will follow from picking a good candidate for F (elliptic with positive index). To use the regular value theorem, we might have to perturb to a sufficiently generic situation - for example consider $F^{-1}(\eta)$ for some small η.

If $\mathcal{M}$ were 0-dimensional and compact, then one would be tempted to count $\mathcal{M}$. In the ASD case, one can define an orientation of $\mathcal{M}$, and it turns out that only the signed count (or counting mod 2) will give an invariant. The definition of a signed count can be seen as the pairing

$$\langle \mathcal{M}, \omega \rangle$$

where $\omega \in H^0(\mathcal{M}, \mathbb{Z})$ is the orientation.

For $\dim \mathcal{M} > 0$, to define integer invariants, we can pick cohomology classes $\eta \in H^d(\mathcal{M}, \mathbb{Z})$ and then define the invariants as

$$\langle \mathcal{M}, \eta \rangle.$$

In the case of Seiberg-Witten theory, this is done as follows. One picks a point $m_0 \in M$ and considers the based gauge group $\mathcal{G}_0$, where

$$\mathcal{G}_0 = \left\{ \varphi \in \mathcal{G} \; | \varphi(p) = p \; \forall p \in \pi^{-1}(m_0) \right\}.$$

This gives a short exact sequence

$$1 \to \mathcal{G}_0 \to \mathcal{G} \to G \to 1$$

and so the based moduli space

$$F^{-1}(0)/\mathcal{G}_0 = \mathcal{M}_0 \to \mathcal{M} = F^{-1}(0)/\mathcal{G},$$

Is a principal G-bundle over $\mathcal{M}$. Hence we can use powers of the characteristic classes of this bundle to pick η.

2 Yang Mills functional, ASD equations. Chern-Weil theory. U(1) example.

Thibault Langlais

2.1 Characteristic Classes and Classifying Space

Suppose, as often happens in life, we are given a fibration $Y \to X$, with fibre F. For example a vector bundle or principal bundle from Section 1. One tool for studying these types of fibrations (in particular, to classify them) is through characteristic classes. That is, given a such a fibration, we want to construct elements $c(Y) \in H^*(X)$ that depend on the bundle.

These classes should satisfy a naturally requirement under pullback. That is, if $f : X' \to X$, then $f^*c(Y) = c(f^*Y)$, where f^*Y is the pullback bundle.

Remark 2.1. Characteristic classes are purely topological. In the theory of classifying spaces, one typically works in the category of CW-complexes.

If the type of fibration being studied is sufficiently nice (as in the case of principal bundles) then one can find a classifying space for the fibration. This is a fibration

$$\mathcal{Y} \to \mathcal{B},$$

with fibre F that satisfies the following universal property: if $Y \to X$ is another fibration with fibre F, then there exists a unique $f_Y : X \to \mathcal{B}$ (up to homotopy) such that

$$f_Y^* \mathcal{Y} \cong Y.$$

Given a classifying space, one can then pick a class $c \in H^*(\mathcal{B})$, and take $f_Y^* c$ to define the characteristic classes of Y.

Remark 2.2. Let $H^1(X, G)$ be the 1st cohomology group with coefficients in the group G. This can be defined along the lines of Čech cohomology theory, i.e. by picking an open cover $\mathcal{U} = \{U_i\}$ and constructing collections of local functions $f_{ij} : U_i \cap U_j \to G$ for all i, j, such that $f_{ij} = f_{ji}^{-1}$ and $f_{ij} f_{jk} = f_{ik}$ in G. The Čech cohomology is then defined as the set of such functions, modulo the equivalence relation $\{f_{ij}\} \sim \{g_{ij}\}$ whenever

$$f_{ij} = h_i g_{ij} h_i^{-1},$$

for all i, j, where $h_i : U_i \to G$. The first cohomology is then defined as the inductive limit of $H^1(\mathcal{U}, G)$ when the cover is refined. This is not a group in general, but we have a functor.

$$H^1(\cdot, G) : CW_h \to \mathrm{Set}, X \mapsto H^1(X, G)$$

where CW_h is the category of CW complexes with maps up to homotopy. One way of describing a principal bundle is to give explicit trivialisations on a cover of X. Hence its fairly clear that each element $H^1(X, G)$ corresponds to a principal G-bundle. The theory of classifying spaces is essentially the observation that this functor is representable, so there is a classifying space $\mathcal{B}$ such that there is a natural isomorphism

$$H^1(\cdot, G) \cong [\cdot, \mathcal{B}]$$

where $[X, \mathcal{B}]$ denotes the homotopy classes of maps $X \to \mathcal{B}$.

Example 2.3 (Chern Classes)**.** Let $Y \to X$ be a rank k complex vector bundle. The classifying space for such bundles is G_k^∞, which is the Grassmanian of k-planes in $\mathbb{C}^\infty$. This can be defined by considering the square

$$\begin{array}{ccc} E_k^n & \to & E_k^{n+1} \\ \downarrow & & \downarrow \\ G_k^n & \to & G_k^{n+1} \end{array}$$

where G_n^k is the Grassmanian (space of k planes in $\mathbb{C}^n$) and E_n^k is the tautological vector bundle, where to each plane $p \in G_n^k$, the fibre of the bundle at p is just p. One then takes the inductive limit to get the bundle $E_k^\infty \to G_k^\infty$. It turns out that the bundle $E_k^\infty \to G_k^\infty$ is the classifying space for Hermitian vector bundles. The cohomology of G_k^∞ may be written as

$$H^{2i}(G_k^\infty, \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_k] \tag{5}$$

with $c_i \in H^{2i}(G_k^\infty, \mathbb{Z})$. These are used to define the chern classes.

Example 2.4 ($\mathbb{C}$-line bundles over $\mathbb{CP}^\infty$)**.** To justify (5) we can consider the case for line bundles over $\mathbb{CP}^\infty$. One can prove that

$$H^*(\mathbb{CP}, \mathbb{Z}) = \mathbb{Z}[c]/c^{n+1}$$

where c generates $H^{2i}(\mathbb{CP}^n)$. The proof follows by induction and Poincaré duality.

2.2 Chern-Weil Point of View

If G is a compact Lie group, then as described in Section 1 we can consider principal G-bundles $P \to X$, where X is some manifold. Chern-Weil theory gives an explicit way to construct characteristic classes of P.

We start by picking a connection $A \in \Omega^1(P, \mathfrak{g})$, and consider the curvature $F_A \in \Omega^2(P, \mathfrak{g})$ which we can view as a two form on X taking values in the adjoint bundle $\mathrm{Ad}P \to X$, as in Proposition 1.15. Let $S(\mathfrak{g}^*)^G$ denote the Ad invariant polynomials. These are maps

$$f : \mathfrak{g} \to \mathbb{C}$$

which are of the form $f(v) = \mathrm{Tr}\left(\sum_i a_i \rho(v)^i\right)$ where $\rho : \mathfrak{g} \to GL(V)$ is a representation of V and $f(Ad_g v) = f(v)$ for all $g \in G$. In the case of $G = U(r)$, the characteristic polynomial

$$\xi \mapsto \det\left(\lambda I - \frac{i}{2\pi}\xi\right) = \lambda^k + \lambda^{k-1}c_1(\xi) + \cdots + c_k(\xi)$$

gives generators c_i of $S(\mathfrak{g}^*)^G$. We have the following claim.

Lemma 2.5. Let $p \in S(\mathfrak{g}^*)$. Then $p(F_A)$ is a well defined differential form. Moreover, $dp(F_A) = 0$ and if A' is a different connection, $p(F_A) - p(F_{A'})$ is exact.

Proof. First consider F_A as an element of $\Omega^2(P, \mathfrak{g})$. Then, as F_A is G equivariant, $p(F_A)$ defines a basic two form taking values in the trivial line bundle over P. The bundle associated to the trivial representation is trivial, and so this defines a differential form on X. The fact that $dp(F_A) = 0$ follows from the Bianchi identity and that $p(\rho(\xi)\rho(\eta)) = p(\rho(\eta)\rho(\xi))$. The final claim follows from the identity

$$F_{A+a} = F_A + da + a \wedge a,$$

where A is a connection and $a \in \Omega^1(X, \mathrm{Ad}P)$. $\square$

Hence, given a principal G-bundle P we can associate a cohomology class to p by picking a connection and calculating the cohomology class of $p(F_A)$.

Remark 2.6. There are two limitations of the Chern-Weil approach, compared to the more general approach of classifying spaces. The first is that we can only get classes of even degree. The second is that these quantities are valued in the real cohomology, not the integral cohomology.

We do however have the following.

Proposition 2.7. In real cohomology, the Chern-Weil approach coincides with the topological approach.

Proof. This follows from the naturality properties. For example $c(f^*E) = f^*c(E)$, $c(E_1 \oplus E_2) = c(E_1)c(E_2)$ (where $c = \sum c_i \in H^*(X, \mathbb{R})$) and both theories have the same normalisation

$$\langle c_1(\mathcal{O}(-1) \to \mathbb{CP}^1, [\mathbb{CP}^1] \rangle = -1.$$

□

2.3 Line Bundles

Let X be a complex surface and $\mathcal{L} \to X$ be a holomorphic line bundle with hermitian connection h. A Chern connection ∇ is a connection on $\mathcal{L}$ such that $\nabla h = 0$ and $\nabla^{0,1} h = \partial_L$. If σ is a local holomorphic section and $H = h(\sigma, \sigma)$, we can define a connection A by

$$A = H^{-1}\partial H = \partial \log H$$

We can compute the curvature of A, and it is given by

$$F_A = dA = \overline{\partial}\partial \log H.$$

and so

$$c_1(\mathcal{L}) = \left[\frac{i}{2\pi}\overline{\partial}\partial \log H\right].$$

2.4 Yang-Mills functional and ASD equations on Hermitian Vector Bundles

Throughout this section, let X be a compact, oriented, Riemannian four manifold. The hodge star operator, restricted to two forms, is such that

$$* : \Lambda^2 \to \Lambda^2, *^2 = 1$$

Hence $*$ is diagonalisable and has two eigenspaces, with eigenvalues ± 1. Moreover changing the orientation of X swaps the eigenspaces, so these have the same dimension. Hence applying this fibrewise we have the bundle decomposition

$$\Lambda^2 X \cong \Lambda^2_+ \oplus \Lambda^2_-.$$

and the two forms also decompose. We write this as $\Omega^2(X) = \Omega_+(X) \oplus \Omega_-(X)$. This also induces a splitting on the cohomology

$$H^2(X) = H^2_+(X) \oplus H^2_-(X)$$

which follows from Hodge theory and a Weitzenböck formula. Hence, we have refined betti numbers $b_\pm$ by taking the dimension of these.

Given a hermitian vector bundle $E \to X$, with metric h we can define

$$* : \Omega^p(X, \mathrm{End}(E)) \to \Omega^{4-p}(X, \mathrm{End}(E))$$

via the map $\omega \otimes T \mapsto *\omega \otimes T^*$, where T^* is the h-dual of T.[1] This leads to a decomposition

$$\Omega^2(X, \mathrm{End}(E)) \cong \Omega_+(X, \mathrm{End}(E)) \oplus \Omega_-(X, \mathrm{End}(E)).$$

Definition 2.8. A connection A is anti-self-dual (ASD) if $\pi_+(F_A) = 0$. Note that $\omega \mapsto \frac{1}{2}(*\omega + \omega)$ is the projection $\Omega^2 \to \Omega_+$ so A is anti-self-dual if and only if

$$F_A = - * F_A.$$

[1] Sometimes authors will just consider sections of $\Lambda_+ \otimes \mathrm{Ad}P$ to be the self dual two forms.

Definition 2.9. Let $\omega \in \Omega^2(X, \mathrm{End}(E))$. Then we can define the norm

$$\|\omega\|_{L_2} = \int_X \mathrm{Tr}(\omega \wedge \omega^*)$$

where $\omega \wedge *\omega$ is given by wedging the form part and composing the endomorphisms. The functional

$$A \mapsto \|F_A\|^2_{L^2}$$

is called the Yang-Mills functional.

Proposition 2.10. A connection A is ASD if and only if A is an absolute minimiser of the Yang-Mills functional.

Proof. This follows from a direct calculation. Note that

$$\begin{aligned}
\|F_A\|_{L^2} &= \int \mathrm{Tr}(F_A \wedge F_A^*) \\
&= \int \mathrm{Tr}(F_A^-(F_A^-)^*) + \int \mathrm{Tr}(F_A^+(F_A^+)^*) \\
&= \int \mathrm{Tr}(F_A^- F_A^-) - \int \mathrm{Tr}(F_A^+ F_A^+) \\
&= -2\int \mathrm{Tr} F_A^+ F_A^+ + \int \mathrm{Tr} F_A^2 \\
&= 2\|F_A^+\|^2 + \int \mathrm{Tr} F_A^2.
\end{aligned}$$

The quantity $\int \mathrm{Tr} F_A^2$ is topological, given by $8\pi^2\left(c_2 - \frac{1}{2}c_1\right)$, which proves the result. $\square$

Remark 2.11. We note the following about the ASD equations. First, $F_A^+ = 0$ is conformally invariant since in 4 dimensions, $\mathrm{Tr}(\omega \wedge *\omega)$ is conformally invariant. Second, the ASD condition is gauge invariant, and is a first order PDE which is nonlinear when the gauge group is non-abelian. The linearisation of the map

$$A \mapsto F_A^+$$

when A is ASD is given by $a \mapsto d_A^+$. If $G = SU(k)$, then $\mathrm{Tr}(F_A) = 0$ and so the energy of the ASD instantons is given by $8\pi^2 c_2(E)$. Note that this implies that the possible energies of ASD instantons is quantised, since c_2 and c_1 can be described by elements in the integral cohomology groups.

2.5 Complex Surfaces

To give some examples of ASD instantons we can use some algebraic geometry. Let (X, g) be a Kähler surface and let $(E, h) \to X$ be a hermitian vector bundle.

Proposition 2.12. Let A be a connection on E. A is ASD if and only if F_A has type $(1,1)$ and $F_A \wedge \omega = 0$

Proof. This follows from writing out the ASD equations in local holomorphic coordinates. $\square$

Note that $F_A \in \Omega^{(1,1)}$ implies that $\overline{\partial}_A^2 = 0$, where $\partial_A + \overline{\partial}_A := d_A$. And hence if A is ASD then $\overline{\partial}_A$ is integrable.

Definition 2.13. A holomorphic vector bundle $\xi \to X$ is hermitian Yang-Mills if

$$F_h \wedge \omega = 0 \tag{6}$$

where F_h is the curvature of the connection defined by h.

Remark 2.14. Taking the trace of (6) one obtains a necessary condition, $c_1(\xi) \cup [\omega] = 0 \in H^4(X)$ for the existence of HYM metrics. The Donaldson-Uhlenbech-Yau theorem gives a sufficient condition (slope stability) for the existence.

2.6 Generalisation to other Gauge Groups

In the above we were considering the ASD equations with structure group $G = U(k)$. In practice one tends to use different structure groups. Typically $SO(3)$ or $SU(2)$. For compact Lie groups we can pick a Ad_G-invariant inner product on the Lie algebra, so we have an Ad_G invariant quadratic form with which to define characteristic classes. In the $SO(3)$ case, the relevant characteristic classes is the Pontryagin classes

$$p_i(E) = c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(X).$$

It turns out that $c_{2i+1}(E \otimes \mathbb{C}) = 0$. Since $SU(2) = \mathrm{Spin}(3)$, the double cover of $SO(3)$, connections on $SU(2)$ bundles are locally the same as connections on $SO(3)$ bundles since the lie algebras are the same.

2.7 Interpretations of the ASD condition

We finish this section by looking at different ways to interpret the ASD condition. Note that the formal adjoint of d_A is given by

$$d_A^* = *d_A * .$$

Hence, if A is ASD,

$$d_A F_A = 0 = d_A^* F_A$$

since $F_A = *F_A$. These are Maxwell's equations. Another way of seeing this is that the Maxwell's equations are the Euler Lagrange equations for the Yang-Mills functional. Hence if $G = U(1)$, then ASD instantons are solutions to Maxwell's equations, and for other gauge groups, one gets solutions to non-abelian gauge theories, for example the standard model takes $G = SU(3) \times SU(2) \times U(1)$. Here the connections A represent gauge bosons, so in the $U(1)$ case the connections are photons.